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1988 J. Phys. A: Math. Gen. 21 1549

(http://iopscience.iop.org/0305-4470/21/7/017)

J. Phys. A: Math. Gen. 21 (1988) 1549-1552. Printed in the UK

A note on the super AKNS equations

Li Yi-shen† and Zhang Li-ning‡
International Centre for Theoretical Physics, Miramore, Trieste, Italy

Received 8 September 1987, in final form 23 November 1987

Abstract. We find some relationships between the usual AKNS scheme with the super one, when its elements take values from the Grassmann algebra on a two-dimensional vector space. The solutions of these super AKNS equations are discussed.

In [1], a hierarchy of super AKNS equations and its Lax pair is given in the case of polynomials in ξ to third order. The super AKNS equations are (replacing β by $-\beta$)

$$q_t = d_0[\tfrac{1}{4}q_{xxx} - \tfrac{3}{2}qrq_x - 3q\alpha\beta_x - 3q\beta\alpha_x - 3(\alpha\alpha_x)_x] + d_1(-\tfrac{1}{2}q_{xx} + q^2r - 2q\alpha\beta + 2\alpha\alpha_x) + d_2q_x - 2d_3q \tag{1}$$

$$r_t = d_0[\tfrac{1}{4}r_{xxx} - \tfrac{3}{2}qrr_x + 3r\alpha\beta_x + 3r\beta\alpha + 3(\beta\beta_x)_x] + d_1(\tfrac{1}{2}r_{xx} - qr^2 + 2r\alpha\beta + 2\beta\beta_x) + d_2r_x + 2d_3r \tag{2}$$

$$\alpha_t = d_0[\alpha_{xxx} - \tfrac{3}{2}qr\alpha_x - \tfrac{3}{4}q_xr\alpha - \tfrac{3}{4}qr_x\alpha + \tfrac{3}{4}q_{xx} + \tfrac{3}{2}q_x\beta_x] + d_1(-\alpha_{xx} + \tfrac{1}{2}\alpha qr - q\beta_x - \tfrac{1}{2}\beta q_x) + d_2d_x - d_3\alpha \tag{3}$$

$$\beta_t = d_0[\beta_{xxx} - \tfrac{3}{2}qr\beta_x - \tfrac{3}{4}q_xr\beta - \tfrac{3}{4}qr_x\beta + \tfrac{3}{4}\alpha r_{xx} + \tfrac{3}{2}r_x\alpha_x] + d_1(\beta_{xx} - \tfrac{1}{2}\beta qr + r\alpha_x + \tfrac{1}{2}\alpha r_x) + d_2\beta_x + d_3\beta \tag{4}$$

where q, r are even elements; $P(q) = P(r) = 0$, α, β are odd elements; $P(\alpha) = P(\beta) = 1$. q, r, α, β are functions of x, t and d_0, d_1, d_2 and d_3 are functions of t.

If q, r, α, β are given values from the Grassmann algebra [2] on a two-dimensional vector space $V = \{e_1, e_2\}$

$$\begin{aligned} q &= q_1 + q_2e_1 \wedge e_2 \qquad & r &= r_1 + r_2e_1 \wedge e_2 \\ \alpha &= \alpha_1e_1 + \alpha_2e_2 \qquad & \beta &= \beta_1e_1 + \beta_2e_2 \end{aligned} \tag{5}$$

where q_1, q_2, r_1, r_2, α_1, α_2, β_1, β_2 are some usual unknown functions, then (1)-(4) reduce to the following equations:

$$q_{1t} = d_0(\tfrac{1}{4}q_{1xxx} - \tfrac{3}{2}q_1r_1q_{1x}) + d_1(-\tfrac{1}{2}q_{1xx} + q_1^2r_1) + d_2q_{1x} - 2d_3q_1 \tag{6}$$

$$r_{1t} = d_0(\tfrac{1}{4}r_{1xxx} - \tfrac{3}{2}q_1r_1r_{1x}) + d_1(\tfrac{1}{2}r_{1xx} - q_1r_1^2) + d_2r_{1x} + 2d_3q_1 \tag{7}$$

$$\begin{aligned} \alpha_{it} = {} & d_0(\alpha_{ixxx} - \tfrac{3}{2}q_1r_1\alpha_{ix} - \tfrac{3}{4}q_{1x}r_1\alpha_i - \tfrac{3}{4}q_1r_{1x}\alpha_i - \tfrac{3}{4}q_{1xx}\beta_i - \tfrac{3}{2}q_{1x}\beta_{ix}) \\ & + d_1(-\alpha_{ixx} + \tfrac{1}{2}q_1r_1\alpha_{ix} + q_1\beta_{ix} + \tfrac{1}{2}q_{1x}\beta_i) \\ & + d_2\alpha_{ix} - d_3\alpha_i \qquad i = 1, 2 \end{aligned} \tag{8}$$

† Permanent address: Department of Mathematics, University of Science and Technology, Hefei, Anhui, People's Republic of China.
‡ Permanent address: Institute of Theoretical Physics, Academia Sinica, Beijing, People's Republic of China.

$$\beta_{it} = d_0(\beta_{ixxx} - \tfrac{3}{2}q_1r_1\beta_{ix} - \tfrac{3}{4}q_{1x}r_1\beta_i - \tfrac{3}{4}q_1r_{1x}\beta_i - \tfrac{3}{4}r_{1xx}\alpha_i - \tfrac{3}{2}r_{1x}\alpha_{ix})$$
$$+ d_1(\beta_{ixx} - \tfrac{1}{2}q_1r_1\beta_{ix} - r_1\alpha_{ix} - \tfrac{1}{2}r_{1x}\alpha_i)$$
$$+ d_2\beta_{ix} + d_3\beta_i \qquad i = 1, 2 \tag{9}$$

$$q_{2t} = d_0[\tfrac{1}{4}q_{2xxx} - \tfrac{3}{2}(q_1r_2 + q_2r_1)q_{1x} - \tfrac{3}{2}q_1r_1q_{2x} + 3q_1(\alpha_1\beta_{2x} - \alpha_2\beta_{1x}) + 3q_1(\beta_1\alpha_{2x} - \beta_2\alpha_{1x})]$$
$$+ d_1[-\tfrac{1}{2}q_{2xx} + 2r_1q_1q_2 + r_2q_1^2 - 2q_1(\alpha_1\beta_2 - \alpha_2\beta_1)$$
$$+ 2(\alpha_1\alpha_{2x} - \alpha_2\alpha_{1x})] + d_2q_{2x} - 2d_3q_2 \tag{10}$$

$$r_{2t} = d_0[\tfrac{1}{4}r_{2xxx} - \tfrac{3}{2}(q_1r_2 + q_2r_1)r_{1x} - \tfrac{3}{2}q_1r_1r_{2x} - 3r_1(\alpha_1\beta_{2x} - \alpha_2\beta_{1x}) - 3r_1(\beta_1\alpha_{2x} - \beta_2\alpha_{1x})]$$
$$+ d_1[\tfrac{1}{2}r_{2xx} - 2q_1r_1r_2 - r_1^2q_2 - 2r_1(\alpha_1\beta_2 - \alpha_2\beta_1)$$
$$+ 2(\beta_1\beta_{2x} - \beta_2\beta_{1x})] + d_2r_{2x} + 2d_3r_2. \tag{11}$$

We have eight equations here but (α_1, α_2) and/or (β_1, β_2) satisfy the same equations, respectively. Note that (6) and (7) for q_1, r_1 are nothing but the usual AKNS equation associated with the Lax pair [3-5] (linearised equations):

$$\varphi_{1x} = -\xi\varphi_1 + q_1\varphi_2 \qquad \varphi_{2x} = r_1\varphi_1 + \xi\varphi_2 \tag{12}$$
$$\varphi_{1t} = A\varphi_1 + B\varphi_2 \qquad \varphi_{2t} = C\varphi_1 - A\varphi_2 \tag{13}$$

where

$$A = d_0[\tfrac{1}{4}(q_1r_{1x} - r_1q_{1x}) + \tfrac{1}{2}q_1r_1\xi - \xi^2] + d_1(\tfrac{1}{2}q_1r_1 - \xi^2) - d_2\xi - d_3$$
$$B = d_0[(\tfrac{1}{4}q_{1xx} - \tfrac{1}{2}q_1^2r_1 - \tfrac{1}{2}q_{1x}\xi + q_1\xi^2)] + d_1(-\tfrac{1}{2}q_{1x} + q_1\xi) + d_2q_1 \tag{14}$$
$$C = d_0(\tfrac{1}{4}r_{1xx} - \tfrac{1}{2}q_1r_1^2 + \tfrac{1}{2}r_{1x}\xi + r_1\xi^2) + d_1(\tfrac{1}{2}r_{1x} + r_1\xi) + d_2r_1.$$

When we find a solution for (q_1, r_1) of (6) and (7) and substitute it into (8) and (9), then (8) and (9) are linear equations for (α_i, β_i).

Using (12) and eliminating the parameter ξ in (13), then (13) can be written in the alternative form

$$\varphi_{1t} = A\varphi_1 + B\varphi_2 = d_0(\varphi_{1xxx} - \tfrac{3}{2}q_1r_1\varphi_{1x} - \tfrac{3}{4}q_{1x}r_1\varphi_1 - \tfrac{3}{4}q_1r_{1x}\varphi_1 - \tfrac{3}{4}q_{1xx}\varphi_2 - \tfrac{3}{2}q_{1x}\varphi_{2x})$$
$$+ d_1(-\varphi_{1xx} + \tfrac{1}{2}q_1r_1\varphi_1 + q_1\varphi_{2x} + \tfrac{1}{2}q_{1x}\varphi_2) + d_1\varphi_{1x} - d_3\varphi_1 \tag{15}$$

$$\varphi_{2t} = C\varphi_1 - A\varphi_2 = d_0(\varphi_{2xxx} - \tfrac{3}{2}q_1r_1\varphi_{2x} - \tfrac{3}{4}q_1r_{1x}\varphi_2 - \tfrac{3}{4}q_{1x}r\varphi_2 - \tfrac{3}{4}r_{1xx}\varphi_1 - \tfrac{3}{2}r_{1x}\varphi_{1x})$$
$$+ d_1(\varphi_{2xx} - \tfrac{1}{2}q_1r_1\varphi_2 - \tfrac{1}{2}r_{1x}\varphi_1 - r_1\varphi_{1x}) + d_2\varphi_{2x} + d_3\varphi_2. \tag{16}$$

It is amusing that (8) and (9) for (α_i, β_i) coincide with (15) and (16) for (φ_1, φ_2). It can be concluded that the solutions of (8) and (9) are nothing but the eigenfunctions of the usual Lax pairs (12) and (13).

When $\alpha_1 = c\alpha_2$, $\beta_1 = c\beta_2$ (c is an arbitrary constant), (10) and (11) for q_2, r_2 reduce to a coupled linear homogeneous equation:

$$q_{2t} = d_0(\tfrac{1}{4}q_{2xxx} - \tfrac{3}{2}(q_1r_2 + q_2r_1)q_{1x} - \tfrac{3}{2}q_1r_1q_{2x})$$
$$+ d_1(-\tfrac{1}{2}q_{2xx} + 2r_1q_1q_2 + r_2q_1^2) + d_2q_{2x} - 2d_3q_2$$
$$r_{2t} = d_0[\tfrac{1}{4}r_{2xxx} - \tfrac{3}{2}(q_1r_2 + q_2r_1)r_{1x} - \tfrac{3}{2}q_1r_1r_{2x}]$$
$$+ d_1(\tfrac{1}{2}r_{2xx} - 2q_1r_1r_2 - r_1^2q_2) + d_2r_{2x} + 2d_3r_2. \tag{17}$$

The coupled equation (17) is nothing but the temporal evolution equation of the linearised equation for the usual AKNS equations (6) and (7). We can conclude that the solutions of (17) for (q_2, r_2) are the symmetries of the usual AKNS equation [6].

By means of cumbersome but straightforward calculations, we have proved that the squared eigenfunctions $(\varphi_1^2, \varphi_2^2)$ of the Lax pair (12) and (13) are the solutions of (17) for q_2, r_2. In the general case, $\alpha_1 \neq c\alpha_2$ or $\beta_1 \neq c\beta_2$, we can use the completeness of the square generalised eigenfunctions of the AKNS system [7] to give the solution for (q_2, r_2) of the coupled inhomogeneous equations (10) and (11). We shall publish this elsewhere.

As an example, we take $d_1 = d_2 = d_3 = 0$, $d_0 = 4$, $\beta = 0$ and $r = -1$. (1)-(4) reduce to the super KdV equation

$$\begin{aligned} q_t &= q_{xxx} + 6qq_x - 12\alpha\alpha_{xx} \\ \alpha_t &= 4\alpha_{xxx} - 6q\alpha_x + 3q_x\alpha. \end{aligned} \tag{18}$$

(6)-(11) reduce to

$$q_{1t} = q_{1xxx} + 6q_1q_{1x} \tag{19}$$

$$\alpha_{it} = 4\alpha_{ixxx} + 6q_1\alpha_{ix} + 3q_{1x}\alpha_i \qquad i = 1, 2 \tag{20}$$

$$q_{2t} = q_{2xxx} + 6q_1q_{2x} + 6q_2q_{1x} - 12(\alpha_1\alpha_{2xx} - \alpha_2\alpha_{1xx}). \tag{21}$$

Similar equations have been deduced in [8]. Equation (19) is the famous KdV equation, its Lax pair is as follows:

$$\varphi_{2xx} = (\xi^2 - q_1)\varphi_2 \qquad \varphi_{2t} = 4\varphi_{2xxx} + 6q_1\varphi_x + 3q_{1x}\varphi_2. \tag{22}$$

It is well known that the one-soliton solution of the KdV equation (19) is

$$q_1 = 2k^2 \operatorname{sech}^2(kx + 4k^3t).$$

The eigenfunction of the Lax pair is

$$\varphi = (c_1\varphi_1 + c_2\varphi_2) \tag{23}$$

where c_1, c_2 are arbitrary constants, and φ_1, φ_2 are as follows:

$$\begin{aligned} \varphi_1 &= [\xi - k \tanh(kx + 4k^3t)] \exp(\xi x + 4\xi^3 t) \\ \varphi_2 &= [\xi + k \tanh(kx + 4k^3t) \exp(-\xi x - 4\xi^3 t). \end{aligned} \tag{24}$$

Solution (23) does not belong to $L^2(-\infty, \infty)$. If we want φ to tend to zero quickly, when $|x| \to \infty$, we take $\xi \to k$ and then

$$\lim_{\xi \to k} \varphi_1 = \lim_{\xi \to k} \varphi_2 = k \operatorname{sech}(kx + 4k^3t). \tag{25}$$

We obtain the soliton-like solution as follows:

$$\alpha_i = k \operatorname{sech}(kx + 4k^3t) \qquad i = 1, 2. \tag{26}$$

In this case, (21) reduces to the linearised form of the KdV equation.

There are two sets of symmetries [9, 10]: the K symmetry,

$$K_l = \phi^l q_{1x} \left(\phi = D^2 + 4q_1 + 2q_{1x}D^{-1}, D = \partial/\partial x, D^{-1} = \int_{-\infty}^{x} \mathrm{d}y \right)$$

when q_1 is one-soliton solution, and K_l is linear dependent with

$$q_2 = q_{1x} = -4k^3 \operatorname{sech}^2(kx + 4k^3t) \tanh(kx + 4k^3t) \tag{27}$$

and the τ symmetry,

$$\tau_n = \phi^n \tau_0 \qquad \tau_0 = 3tq_{1x} + \tfrac{1}{2}. \tag{28}$$

Only two symmetries

$$\tau_0 = 3tq_{1x} + \tfrac{1}{2} \qquad \tau_1 = 3tq_{1t} + xq_{1x} + 2q_1$$

are local, but τ_0 does not tend to zero as $|x| \to \infty$. We obtain one local solition-like solution

$$q_2 = \tau_1 = 4k^2\,\mathrm{sech}^2(kx + 4k^3t)[1 - (4xk + 12tk^3)\tanh(kx + 4k^3t)]. \qquad (29)$$

To conclude we list some problems: how do we prove the above results for all super AKNS hierarchies; do the same results appear in another integrable super system; and when we take values from the Grassmann algebra on the n-dimensional vector space, what will happen?

Acknowledgments

The authors would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

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